ВЕСТНИК ГОСУДАРСТВЕННОГО УНИВЕРСИТЕТА МОРСКОГО И РЕЧНОГО ФЛОТА ИМЕНИ АДМИРАЛА С. О. МАКАРОВА

# АВТОМАТИЗАЦИЯ И УПРАВЛЕНИЕ ТЕХНОЛОГИЧЕСКИМИ ПРОЦЕССАМИ И ПРОИЗВОДСТВАМИ

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## OPTIMIZATION OF SHIP DYNAMIC SYSTEMS AND TECHNOLOGICAL PROCESSES IN WATER TRANSPORT USING SYMBOLIC COMPUTING IN MATLAB

**S. O. Baryshnikov, V. V. Sakharov, A. A. Chertkov**

Makarov State University of Maritime and Inland Shipping,
St. Petersburg, Russian Federation

*The study aims to increase the efficiency and accuracy of solving optimal control problems for ship dynamic systems and technological processes in water transport under conditions of digital transformation using symbolic computing tools. The paper addresses the problem of optimal control of a nonlinear dynamic object by representing the system in symbolic mathematical form. The proposed computational algorithm provides an analytical solution of differential equations through linearization and integration in a standard matrix representation. Using the Hamiltonian approach, which ensures the transition from functional minimization to static optimization, a control vector is derived and the system of equations is transformed into symbolic form. Taking into account the syntax of symbolic functions, an analytical block describing system dynamics is identified, and a solver is constructed that includes the system dynamics and boundary conditions for state variables at the initial and final moments of the solution interval. As a result, equations for state and control variables are obtained, which can subsequently be converted into numerical form for quantitative evaluation and graphical interpretation. Using MATLAB programs, estimates of four boundary conditions are obtained and presented graphically. The proposed algorithmic solution of the boundary value problem differs from existing approaches by employing an analytical model expressed in symbolic terms. A discrete analogue of the model is obtained on the basis of the A. N. Krylov matrix with norm estimation and control representation in CVX format. The results confirm the correctness of the developed algorithms and software and demonstrate the expediency of combining analytical and numerical methods for modeling and optimization of dynamic systems.*

*Keywords: optimal control; symbolic computing; ship dynamic systems; technological processes; Hamiltonian approach; boundary value problem; Krylov matrix; MATLAB; dynamic system modeling*

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## ОПТИМИЗАЦИЯ СУДОВЫХ ДИНАМИЧЕСКИХ СИСТЕМ И ТЕХНОЛОГИЧЕСКИХ ПРОЦЕССОВ НА ВОДНОМ ТРАНСПОРТЕ С ПРИМЕНЕНИЕМ СИМВОЛЬНЫХ ВЫЧИСЛЕНИЙ В СРЕДЕ MATLAB

**С. О. Барышников, В. В. Сахаров, А. А. Чертков**

ФГБОУ ВО «ГУМРФ имени адмирала С. О. Макарова»,
Санкт-Петербург, Российская Федерация

*Темой исследования является повышение эффективности и точности решения задач оптимального управления технологическими процессами и производствами на водном транспорте в условиях цифровой* *трансформации с применением инструментов символьной математики. В работе решена задача оптимального управления нелинейным динамическим объектом средствами цифровизации в кодах символьной математики. Предложенный вычислительный алгоритм предусматривает аналитическое решение дифференциальных уравнений путем линеаризации и интегрирования их в стандартном матричном формате. Согласно гамильтониану, обеспечивающему переход от функциональной минимизации к статической оптимизации, получен вектор управления, а также обеспечен перевод системы уравнений в символьный формат. С учетом синтаксиса функций выделен блок динамики системы в аналитическом виде, а также образован решатель, состоящий из блока динамики и граничных условий на переменные состояния в начале и по окончании времени решения. В результате составлены уравнения для переменных состояния и управления, которые для количественных оценок и графической интерпретации свободно переводятся в числовой формат. С помощью программ в кодах MATLAB выполнены оценки четырех краевых условий, приведенных на графиках. Отличие предложенного алгоритмического решения краевой задачи от существующих решений состоит в применении аналитической модели в символьных терминах. Дискретный аналог модели получен на базе матрицы А. Н. Крылова с оценкой нормы и управления в формате CVX. Приведенные решения позволяют сделать вывод о корректности представленных алгоритмов и программ, а также о целесообразности применения для моделирования систем аналитических методов в сочетании с численными.*

*Ключевые слова: оптимизация, алгоритм, гамильтониан, краевые условия, символьный формат вычислений, аналитическое решение, матрица А. Н. Крылова, моделирование.*

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**(Introduction)**

Many practical optimization problems arising in the modeling of dynamic control processes of shipboard objects and production systems in water transport require adaptation to specific models using basic computational systems [1]. As computational tools are regularly updated, new algorithms and software have emerged whose computational properties have significantly improved in terms of solution speed and reliability for applied problems, which has created a need to interpret these new solutions and to develop methods for their effective application [2]. In this context, special attention should be given to the MATLAB computational environment, which is one of the most powerful universal integrated systems of computer mathematics [3]. Its toolboxes, in addition to existing computational tools, have been supplemented with solvers in CVX format capable of working with models containing linear matrix inequalities, which has not only expanded the class of solvable mathematical programming problems but has also led to new approaches to the synthesis of closed-loop systems possessing asymptotic stability [2]. Particular attention should be paid to the potential capabilities of the symbolic computing package (Symbolic Math Toolbox), on the basis of which a computational algorithm for the synthesis of an optimal dynamic system has been developed.

Symbolic computations are referred to as analytical, since the initial data for computations are specified in the form of analytical functional relationships, and the computation results are presented in symbolic (analytical) form [4]. When boundary conditions and numerical values of parameters of the systems under study are specified, the obtained solutions correspond to particular cases of the results of symbolic computations. The file system of the Symbolic Math Toolbox imparts qualitatively new properties to the process of automating complex analytical computations and performing mathematical transformations. This makes it possible to simplify complex functional relationships consisting of polynomial, trigonometric, exponential, logarithmic components, and others [5].

When modeling dynamic processes, numerical optimization methods based on iterative algorithms are used in combination with analytical algorithms that include mathematical transformations and routine operations requiring computational automation by means of symbolic mathematics. Numerical and analytical computations that lead to identical results in modeling are not only useful but, in many applications, necessary [6]. When modeling dynamic systems and algorithmizing computational processes using the Symbolic Math Toolbox, it is necessary to satisfy the requirements related to the syntax of package functions, the creation of symbolic objects, and the transition to the numerical domain, among others. The proposed optimization algorithm, its implementation method, and the performed calculations are aimed at partially solving this problem.

Symbolic computations in control problems make it possible to obtain direct analytical relationships between controller parameters, object parameters, and desired models of closed-loop systems, while reducing the labor intensity of analytical transformations in the process of modeling and searching for optimal solutions [7]. Due to the complexity of analytical transformations, such solutions are often implemented for systems of relatively low dimensionality, while parameters that are less subject to variation are assigned in numerical form. As a result, polynomial equations take on a combined analytical and numerical form. Certain operations can be performed in the presence of formulas with symbolic coefficients and completed by numerical calculations with graphical representations of implicitly defined functions. In such cases, analytical solutions can be obtained using symbolic tools.

## Methods and Materials

The control problem of a nonlinear dynamic system consists in finding the minimum:

$$J = \Psi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t),\ t)dt \tag{1}$$

for the system

$$\dot{x}(t) = f(t, x(t), u(t)),\ t \in [t_0, t_f],\ x(t_0) = x_0, \tag{2}$$

where $x(t) \in R^n$ — the state; $u(t) \in R^m$ — the control; $L(x, u, t)$ — the running values of the performance criterion (e.g., operating costs); $\Psi(x(t_f), t_f)$ — the terminal cost at the final time instant (a scalar quantity).

Assume that the function f(t,x(t),u(t))f(t, x(t), u(t))f(t,x(t),u(t)) is continuous with respect to the state and control variables, and that there exists a control u(t)=u∗(t)u(t) = u^*(t)u(t)=u∗(t) that transfers the system from the given initial state x(t0)x(t_0)x(t0) to the terminal state x(tf)x(t_f)x(tf) while satisfying the necessary optimality conditions.

If linearization of equations (1) and (2) is admissible, and it is possible to construct an adequate (within specified limits) matrix model in the state space, then for time-invariant systems the tools for automating computations in the time and frequency domains can be applied. In this case, the modeling process is significantly simplified due to efficient methods for integrating differential equations in a standard matrix form [8]. Thus, for state-space models, the optimization problem can be formulated in the following form:

$$\min J = \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (x^T(t) Q x(t) + u^T R u(t)) dt; \tag{3}$$

$$\dot{x}(t) = Ax(t) + Bu(t);$$
$$x(0) = x_0,\ \ u(t) = K,\ \ t \in [t_0,\ t_f],\ \ t_0 = 0, \tag{4}$$

where $x(t)$ и $u(t)$ are piecewise-continuous state and control vectors, respectively.

The state and control matrices $A$ и $B$ have appropriate dimensions, $x(0)$ — denotes the initial state. The set $K \subseteq R^m$ is closed. The terminal time $t_f$ is specified, for which the terminal state $x(t_f)$ may also be prescribed. The matrices $Q, S \in R^{n \times m}$ are positive semidefinite.

It is now possible to formulate the necessary optimality conditions for the minimization criterion (3). The key point in deriving the necessary conditions is that, by means of the Hamiltonian, the functional minimization problem for the function H(x,p,u,t)H(x,p,u,t)H(x,p,u,t) is transformed into a static optimization problem [9]. Taking into account the system dynamics (4), the Hamiltonian is constructed as follows:

$$H(x(t),\ p(t),\ u(t),\ t) = \frac{1}{2} x^T Q x(t) + \frac{1}{2} u^T R u(t) + p^T [Ax(t) + Bu(t)], \tag{5}$$

где $p(t)$ — the adjoint state variable.

As a result, based on equation (5), we obtain:

$$\dot{x}(t) = \frac{\partial H}{\partial p(t)} = Ax(t) + BR^{-1} B^T p(t); \tag{6}$$

$$\dot{p}(t) = -\frac{\partial H}{\partial x(t)} = -Qx(t) - A^T p(t); \tag{7}$$

$$u(t) = \arg\ \min\ H(x(t),\ p(t),\ u(t),\ t),\ u(t) \in K,\ 0 \le t \le t_f]; \tag{8}$$

$$p(t_f) = Sx(t_f),\ \ x(0) = x_0. \tag{9}$$

Equations (6)–(9) represent a two-point boundary value problem, in which the initial value of the state $x(t)$ is $x(0) = x_0$, and the corresponding condition for the adjoint variable is $p(t)$ $-p(t_f) = Sx(t_f)$. Its solution is obtained in symbolic mathematics format.

The application of specific optimization methods requires appropriate mathematical support [10]. The Symbolic Math Toolbox is intended for performing symbolic computations, which is possible provided that the syntax of the functions used is satisfied [11]. Symbolic variables in MATLAB are declared using the **`syms`** operator, followed by a list of identifiers separated by spaces, which are converted into symbolic variables.

The properties of symbolic variables are taken into account when performing analytical transformations of systems of equations and inequalities written using them. To convert numerical or string values of functions into symbolic expressions, the **`syms`** function is used. When analytically solving a system of equations in symbolic form, it should be represented as a one-dimensional array whose elements are individual equations.

Taking the above into account, an analytical solution of the control problem for a dynamic object based on the minimum control energy criterion is considered in the Symbolic Math Toolbox format and compared with a numerical solution in the CVX environment [12]. For definiteness of computations, a dynamic object in the state space with the following matrices is selected:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},\ \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ \ D = 0. \tag{10}$$

Assume that it is required to minimize the performance criterion:

$$J(u) = \int_{t_0}^{t_f} \frac{1}{2} u^2(t)dt,$$

subject to the constraints defined by object (10) under the specified boundary conditions:

$$x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T,\ \ x(2) = \begin{bmatrix} 5 & 2 \end{bmatrix}^T,\ \ t \in [t_0,\ t_f],\ \ t_0 = 0,\ t_f = 2\,c.$$

The solution is obtained using the MATLAB file sah051dd.m, which incorporates symbolic mathematics functions:

```
% sah051dd.m. Символьные вычисления в оптимальном управлении
% 1. Уравнения состояния
syms x1 x2 p1 p2 u t
Dx1 = x2;
Dx2 = -x2 + u;
% 2. Текущее значение критерия качества – символьный объект
syms g;
g = 1/2*u^2;
% 3. Гамильтониан
H = g + p1*Dx1 + p2*Dx2;
% 4. Уравнения косостояния
Dp1 = - diff(H,x1);
Dp2 = - diff(H,x2);
% 5. Определение управления u
du = diff(H,u);
```

```
u_u = solve(du,u);
%=======================================
% 6. Подстановка u во второе уравнение состояния
Dx2 = subs(Dx2, u, u_u);
%=======================================
% 7. Необходимые условия оптимальности в терминах символьной математики.
% Введение граничных условий. Интегрирование дифференциальных уравнений.
syms x1(t) x2(t) p1(t) p2(t)
eqns=[diff(x1,t)==x2,diff(x2,t)==-p2-x2,diff(p1,t)==0,diff(p2,t)==p2-p1];
S=dsolve(eqns,x1(0)==0,x2(0)==0,x1(2)==5, x2(2)==2);
%--------------------------------------------------
% 8. Общее решение (без граничных условий) с постоянными интегрирования
S1=dsolve(eqns)
%---------------------------------------------------
% Графические построения
t1=0:0.5:2;
X1=S.x1;
X2=S.x2;
P_2=S.p2
fplot(X1,X2,[0 2]),grid
```

The file consists of several sections:

```
% sah051dd.m. Symbolic computations in optimal control
% 1. State equations
syms x1 x2 p1 p2 u t
Dx1 = x2;
Dx2 = -x2 + u;

% 2. Current value of the performance index as a symbolic object
syms g;
g = 1/2*u^2;

% 3. Hamiltonian
H = g + p1*Dx1 + p2*Dx2;

% 4. Costate equations
Dp1 = - diff(H,x1);
Dp2 = - diff(H,x2);

% 5. Determination of the control u
du = diff(H,u);
u_u = solve(du,u);

%=======================================
% 6. Substitution of u into the second state equation
Dx2 = subs(Dx2, u, u_u);
%=======================================
% 7. Necessary optimality conditions in terms of symbolic mathematics.
%    Specification of boundary conditions. Integration of differential
% equations.
syms x1(t) x2(t) p1(t) p2(t)
eqns=[diff(x1,t)==x2,diff(x2,t)==-p2-x2,diff(p1,t)==0,diff(p2,t)==p2-p1];
S=dsolve(eqns,x1(0)==0,x2(0)==0,x1(2)==5, x2(2)==2);

%--------------------------------------------------
```

```
% 8. General solution (without boundary conditions) with integration
% constants
S1=dsolve(eqns)

%----------------------------------------------------
% Graphical representations
t1=0:0.5:2;
X1=S.x1;
X2=S.x2;
P_2=S.p2
fplot(X1,X2,[0 2]),grid
```

The file consists of several sections. In the first section, symbolic variables are introduced and the system dynamic equations are formulated. In the second section, a symbolic object is introduced representing the current value of the performance index (the integrand of the cost functional). The third section presents the Hamiltonian, while the fourth section contains the differential equations of the costate variables. The fifth and sixth sections are intended for the formation of the expression Dx2D x_2Dx2.

The introduction of the necessary optimality conditions makes it possible to proceed to the solution of a two-point boundary value problem. For this purpose, in the seventh section, symbolic objects are introduced and a vector containing the system of differential equations is formed. The solution of this system is obtained using the dsolve function under the specified boundary conditions. The analytical solutions of the problem are represented by the array S. The symbolic equations extracted from the array *S*

X1 = S.x1, X2 = S.x2 и P2 = S.p2 по схеме X1 = vpa(X1,5) have the following form:

$$X1 = 7.2918*t + 6.6983*\exp(-1.0*t) - 0.59348*\exp(t) - 6.1048;$$

$$X2 = 7.2918 - 0.59348*\exp(t) - 6.6983*\exp(-1.0*t);$$

$$P2 = 1.18696*\exp(t) - 7.29179.$$

Since the control is given $u = -p_2$, we obtain

$$u = 7.29179 - 1.18696*\exp(t).$$

The general solution is represented by the array $S_1$. In the absence of boundary conditions, the state equations contain integration constants $C_1$, $C_2$, $C_3$ и $C_4$, which are subject to determination. After extracting these equations from the array $S_1$ the following expressions are obtained:

$$X1 = (C_3*\exp(t))/2 - C_4*t - C_1 - (C_2*\exp(-t)) / 2;$$

$$X2 = (C_3*\exp(t))/2 - C_4 + (C_2*\exp(-t))/2.$$

Let us introduce the vector $C = [C_1\ C_2\ C_3\ C_4]'$. Then, the following equations can be used to determine the integration constants:

$$X1 = [-1/2, -\exp(-t)/2, \exp(t)/4, -t/2] * C; \quad (11)$$

and

$$X2 = [\exp(-t) / 2, \exp(t)/4, -1/2] * [C_2\ C_3\ C_4]'. \quad (12)$$

Equation (11) allows determining all components of the vector C. According to equation (12), only the components subject to evaluation are $C_2$ $C_3$ и $C_4$. Для оценки сформируем матрицы To perform the evaluation, the following matrices are formed $H_1$ и $H_2$. The components of the equations given in parentheses are used to obtain numerical values over the time interval *t* from zero to 2 s. A time step of 0.05 s is used for the computations. As a result, the following matrices are obtained $H_1$ и $H_2$ of appropriate dimensions (41 × 4) and (41 × 3) respectively. For the evaluation, the CVX format is used, which allows, based on the norm norm(H1*C1 – X1′) to obtain numerical values of the vector *C*.

A fragment of the solution in MATLAB code is presented below:

```
t=0:0.05:2;
H1=[]; H2=[];
for t=0:0.05:2;
    h1=[-1/2 -exp(-t)/2 exp(t)/4 t/2];
    h2=[exp(-t)/2 exp(t)/4 1/2];
    H1=[H1;h1];
    H2=[H2;h2];
end
H1; H2;
% Решение в CVX-формате.
cvx_begin
variable C1(4)
minimize(norm(H1*C1-X1',2))
subject to
cvx_end
cvx_begin
variable C2(3)
minimize(norm(H2*C2-X2',2))
subject to
cvx_end
C1;          C2;          x1m=H1*C1;          x2m=H2*C2;
     [X1' x1m X2' x2m]
```

The following solutions are obtained:
according to formula (11)

$$C_{11} = [12.2097 \quad -13.3966 \quad -2.3739 \quad 14.5836];$$

according to formula (12)

$$C_{12} = [-13.3966 \quad -2.3739 \quad 14.5836].$$

The graphs shown in Fig. demonstrate the equivalence of the solutions under the imposed boundary conditions X1(0) = 0, X2(0) = 0, X1(2) = 5, X2(2) = 2.

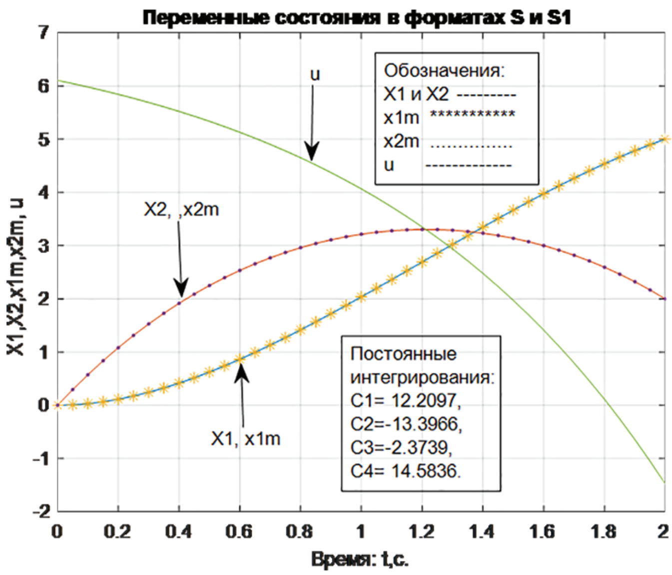


*Fig. 1.* Time evolution of the state variables under the control action X1, X2 и $x_{1m}$, $x_{2m}$ from the arrays *S* и $S_1$ after substituting the integration constants

The file intended for solving optimal control problems with various boundary conditions, which are specified in the line S of the seventh section, was used to obtain analytical functions in symbolic computation format, followed by the construction of the graphs shown in Fig. 2.

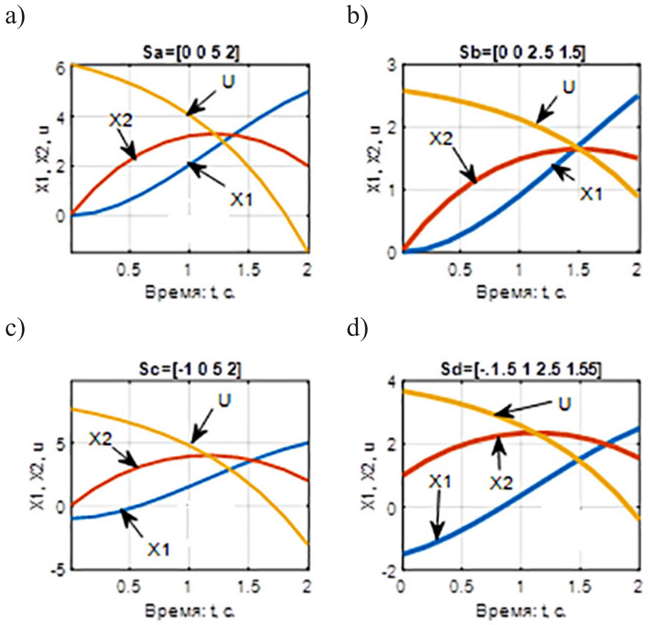


*Fig. 2*. System control under different boundary conditions
*a* — X1(0) = 0; X2(0) = 0; X1(2) = 5; X2(2) = 2; *b* — X1(0) = 0; X2(0) = 0; X1(2) = 2.5; X2(2) = 1.5; *c* — X1(0) = -1; X2(0) = 0; X1(2) = 5; X2(2) = 2; *d* — X1(0) = –1.5; X2(0) = 1; X1(2) = 2.5; X2(2) = 1/55

The main result of the computations is the derivation of analytical solutions for all state and control variables expressed in symbolic form. Below, the solutions obtained for the system with boundary (terminal) conditions are presented

$$X1(0) = -1.5;\ X2(0) = 1;\ x1(2) = 2.\ 5;\ X2(2) = 1.55,$$

which corresponds to Fig. 2(g). The analytical (exact) solutions expressed in the symbols of the Symbolic Math Toolbox are given below:

$$X1 = t*((29*\exp(2))/80 + 131/80) - (84*\exp(2) + 29*\exp(4) - 69)/(80*(\exp(2) - 1)) -$$
$$- (\exp(t)*(29*\exp(2) - 51))/(80*(\exp(2) - 1)) +$$
$$+ (\exp(-t)*\exp(2)*(29*\exp(2) - 7))/(80*(\exp(2) - 1)) \quad (13)$$

$$X2 = (29*\exp(2))/80 - (\exp(t)*(29*\exp(2) - 51))/(80*(\exp(2) - 1)) -$$
$$- (\exp(-t)*\exp(2)*(29*\exp(2) - 7))/(80*(\exp(2) - 1)) + 131/80 \quad (14)$$

$$u = -((\exp(t)*(29*\exp(2) - 51))/(40*(\exp(2) - 1)) - (29*\exp(2))/80 - 131/80). \quad (15)$$

For practical calculations, it is convenient to switch to a numerical format by applying the vpa function with the required number of significant digits specified for visual evaluation. The numerical counterparts of expressions (13)–(15) are given below.:

$$X1 = vpa(X1,5);$$

$$X1 = 4.316*t + 2.9966*\exp(-1.0*t) - 0.31946*\exp(t) - 4.1771; \quad (16)$$

$$X2 = vpa(X2,5);$$

X2 = 4.316 – 0.31946*exp(t) – 2.9966*exp(–1.0*t); (17)

u = vpa(u,5);

u = 4.316 – 0.63892*exp(t). (18)

Identical analytical relationships (13) and (16), (14) and (17), as well as (15) and (18), have been obtained. It should be noted that with an increase in the dimensionality and complexity of the dynamic system, only the exact solution of the problem is of practical interest [13].

**Results**

Discrete control of a ship dynamic system is performed using the A. N. Krylov matrix. For this purpose, a sequence of solutions in a linear time-invariant (LTI) system is considered with a discretization step $T$, that is significantly smaller than the time constants of the controlled object. The state matrix $A_d$ and the control matrix $B_d$ of the discrete-time system are assumed to be known. Under these conditions, the following step-by-step representation is obtained

$$x(1) = A_d \cdot x(0) + B_d \cdot u(0);$$

$$x(2) = A_d \cdot x(1) + B_d \cdot u(1);$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x(n) = A_d \cdot x(n-1) + B_d \cdot u(n-1), \quad n = 0, 1, 2, 3, \ldots, N;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x(N) = A_d \cdot x(N-1).$$

The above equations can be written in matrix form for given initial $x(0)$ and terminal $x(N)$ values of the state variable [1]:

$$x(N) - A_d^N \cdot x(0) = [A_d^{N-1} \cdot B_d,\ A_d^{N-2} \cdot B_d,\ \ldots,\ A_d \cdot B_d,\ B_d] \cdot U, \qquad (19)$$

where the control vector is

$$U = [u(N-1),\ u(N-2),\ \ldots,\ u(1),\ u(0)]^T.$$

Let us introduce the notation

$$G = [A_d^{N-1} \cdot B_d,\ A_d^{N-2} \cdot B_d,\ \ldots,\ A_d \cdot B_d,\ B_d]. \qquad (20)$$

Then, taking into account (19) and (20), the optimal estimate of the vector $U^*$, which ensures minimum energy losses when transferring the object from the initial state to the terminal state, is determined using the Moore–Penrose pseudoinverse:

$$U^* = G^+ \cdot [x(N) - A_d^N \cdot x(0)]. \qquad (21)$$

Since expression (20) represents the A. N. Krylov matrix, which belongs to the gallery of special matrices available in the MATLAB environment, it is introduced into the computational procedure in accordance with the following syntax:

Kr = gallery('krylov', Ad, Bd, N).

The application of the A. N. Krylov matrix in computational environments makes it possible to fundamentally change the modeling process, to construct an optimal estimation system for the state vector variables, and ultimately to use CVX technologies for generating trajectory processes of the state and control variables in the optimal mode.

The MATLAB script sah051hhK.m for solving the problem of optimal discrete-time system synthesis is presented below:

```
% sah051hhK.m.
%=============================================
% 1. Исходные данные
A=[0 1;0 -1]; B=[0 1]'; C=[1 0]; D=0;
sys=ss(A,B,C,D)
A=sys.A; B=sys.B
%=============================================
% 2. Переход к дискретной модели
T=0.1;
[Ad,Bd]=c2d(A,B,T)
% 3. Применение матрицы Крылова для численного решения в формате CVX
N=20;
Kr=gallery('krylov',Ad,Bd,N)
%=================================================
% 4. Оценка вектора w и формирование минимизируемой нормы
s1=inv(Kr*Kr');
s2=s1*Kr;
w1=[5 2]*s2;
w=w1'
%=================================================
% 5. Оценка вектора управления и расчет переменных состояния
% по дискретной модели
m1=N;
cvx_begin
  variable u(m1)
 minimize(norm(u-w,2));
    subject to
       cvx_end
       U=rot90(u,2);
 X=[];
 xa0=[0 0]';
 for p=1:20;
     if p<2
         x=Ad*xa0+Bd*U(1);
     else
         x=Ad*x+Bd*U(p);
     end
     X=[X x];
 end
 X=[xa0 X];
% 6. Графические построения
     k=0:20;
     U=[U(1);U];
         stairs(k,X','LineWidth',3);
     hold on
     stairs(k,U,'LineWidth',2),grid
     xlabel(' k*T,    T=0,1c.')
     ylabel(' X1, X2, U')
     title('Управление с применением матрицы А.Н. Крылова')
     hold off
```



The script is conditionally divided into several sections. *The first section* presents the initial data. In *the second section*, the transition from the continuous-time model to the discrete-time model is performed with a discretization step T=0.1 s. *The third section* specifies the number of steps N=20 and selects the Krylov matrix. In *the fourth and fifth sections*, the estimation of the column vector w is carried out**.** The estimation of this vector represents the most computationally intensive part of the procedure, which is based on solving equation (21).

The estimation of *w* determines the correctness of fitting the elements of the control vector and justifies the introduction of the norm($u - w$) for computing the optimal operating mode within the cvx_begin – cvx_end. In essence, the solution obtained using the least-squares method is reformulated as a convex programming problem, which enables computations to be performed within an iterative loop. This is followed by the computation of the system trajectory in the optimal mode. To ensure the required sequence of the optimal control action applied to the system, a vector rotation operator that rotates the control vector *u* by 180° is used.

*The sixth section* contains operators that define the graphical representations. The *stairs* operator is used, according to which only a limited number of graphical functions can be employed to control line widths. Therefore, graph superposition is implemented using the *hold on* operator, while exiting the superposition mode is ensured by the *hold off* operator. The graphical representation of the solution of the optimal boundary-value problem is shown in Fig. 3.

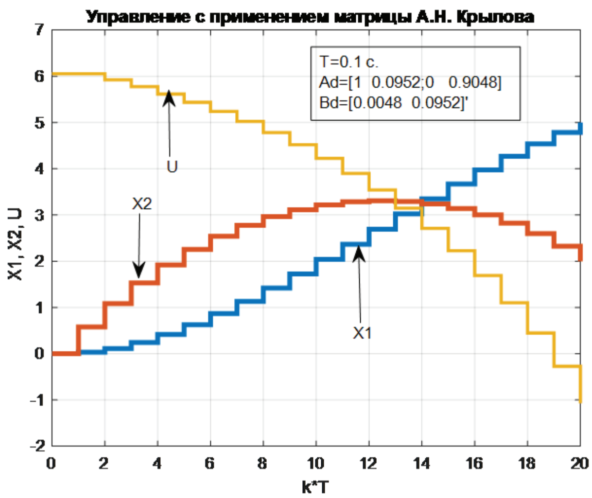


*Fig. 3.* Trajectories of the discrete-time control system

As can be seen from Fig. 3, discrete control of the ship dynamic system in the optimal mode has been implemented. Thus, the application of the A. N. Krylov matrix made it possible to correctly fit the elements of the control vector and to use CVX technologies for the optimal estimation of the state variable vector.

## Discussion

This study considers the application of analytical solutions to optimization problems based on symbolic mathematics tools. The use of new methods and means of digital signal processing employing CVX technologies and solvers intended for convex programming problems contributes to the improvement and further development of optimal control system design technologies [14].

The solutions proposed in this paper regarding the application of analytical methods in control problems of technological processes and production systems in water transport facilities may be of interest to specialists using digital technologies to enhance the efficiency and quality of problem solving.

## Conclusions

Based on the conducted research, the following conclusions can be drawn:

1. Analytical solutions of optimization problems using symbolic mathematics functions constitute a reliable approach to improving the efficiency and quality of designing ship dynamic systems for stabilization, positioning, monitoring, and course control.

2. The application of the A. N. Krylov matrix in digital optimization problems is a necessary condition for improving modeling processes and advancing computer technologies intended for the analysis and synthesis of high-dimensional dynamic systems.

3. The use of CVX technologies for improving computational processes, designed to solve convex programming problems and to synthesize dynamic systems based on linear matrix inequalities adapted to linear and quadratic programming techniques, significantly expands the class of solvable problems within the modern MATLAB computational environment.

4. The presented quantitative estimates of solutions obtained using symbolic computations confirm the correctness of the proposed approaches and demonstrate their high efficiency.

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## APPENDIX

### MATLAB Codes and Figure Explanations

This appendix provides concise explanatory notes for the MATLAB codes and the corresponding figures presented in the paper. Due to the formal programming syntax, the MATLAB scripts are not translated verbatim; instead, their functionality and relation to the graphical results are described.

### MATLAB Codes

The MATLAB script sah051dd.m implements an analytical solution of a continuous-time optimal control problem using the Symbolic Math Toolbox. The script defines the system state equations, formulates the performance index and the Hamiltonian, derives the costate equations, and determines the optimal control law. The resulting two-point boundary value problem is solved symbolically using the dsolve function. Exact analytical expressions for the state, costate, and control variables are obtained and subsequently visualized.

A supplementary code segment applies CVX-based convex optimization to estimate the integration constants of the general symbolic solution. This procedure ensures consistency between symbolic analytical solutions and numerical boundary conditions.

The script sah051hhK.m is used for the synthesis of an optimal discrete-time control system. The continuous model is discretized, and the A. N. Krylov matrix is employed to construct an optimal control sequence. The estimation of the control vector is formulated as a convex optimization problem solved using CVX. The script generates optimal state and control trajectories in discrete time.

**Figure Explanations**

**Figure 1** illustrates the time evolution of the state variables under the action of the optimal control input obtained from the symbolic solution of the continuous-time problem. The figure confirms the consistency of the analytical solution with the imposed boundary conditions.

**Figure 2** presents control and state trajectories for different boundary conditions, demonstrating the flexibility of the symbolic solution framework and the exact analytical dependence of the system response on the prescribed constraints.

**Figure 3** shows the trajectories of the discrete-time control system synthesized using the Krylov matrix and CVX optimization. The figure confirms the correctness of the discrete optimal control formulation and its agreement with the theoretical results.

## FIGURE CAPTIONS (ENGLISH VERSION)

Figure 1.

Time evolution of the state variables under the action of the control input u.

Legend:
$X_1$, $X_2$ — state variables
u — control input
Axes:
x-axis: Time, t (s)
y-axis: $X_1$, $X_2$, u
Text inside the figure:
Integration constants:
$C_1 = 12.2097$
$C_2 = 13.3966$
$C_3 = 2.3739$
$C_4 = 14.5836$

Figure 2.

Control of the dynamic system under different boundary conditions.

Subfigures:
(a) State trajectories $X_1(t)$ and $X_2(t)$
(b) Control input u(t)
(c) Costate variable $p_2(t)$
(d) State trajectories corresponding to alternative boundary conditions

Axes (for all subfigures):
x-axis: Time, t (s)
y-axis: State, costate, or control variables

Note:
The figure illustrates the analytical solutions obtained
using symbolic computations for various bound-ary conditions.

Figure 3.

Trajectories of the discrete optimal control system.

Legend:
$X_1(k)$, $X_2(k)$ — discrete state variables
u(k) — discrete control input

Axes:
x-axis: Discrete time step, k
y-axis: State and control variables

Note:
The figure demonstrates the implementation of optimal discrete control
using the A. N. Krylov matrix and CVX-based optimization.

### Remarks

The presented MATLAB codes serve as computational illustrations of the proposed methods, while the fig-ures provide visual confirmation of the analytical and numerical results. Together, they demonstrate the ef-fectiveness of combining symbolic computations with numerical optimization techniques for solving opti-mal control problems in water transport systems.


**INFORMATION ABOUT THE AUTHORS**

**Baryshnikov Sergey Olegovich** —
Grand PhD in Technical Sciences, professor
Admiral Makarov State University
of Maritime and Inland Shipping
5/7 Dvinskaya Str., St. Petersburg,
198035, Russian Federation
e-mail: *BarychnikovSO@gumrf.ru*
**Saharov Vladimir V.** —
Grand PhD in Technical Sciences, professor,
Admiral Makarov State University
of Maritime and Inland Shipping
5/7 Dvinskaya Str., St. Petersburg,
198035, Russian Federation
e-mail: *saharov_@rambler.ru*
**Chertkov, Alexandr A.** —
Grand PhD in Technical Sciences, associate professor
Admiral Makarov State University
of Maritime and Inland Shipping
5/7 Dvinskaya Str., St. Petersburg,
198035, Russian Federation
e-mail: *chertkov51@mail.ru*,
*kaf _electroprivod@gumrf.ru*

**ИНФОРМАЦИЯ ОБ АВТОРАХ**

**Барышников Сергей Олегович** —
доктор технических наук, профессор
ФГБОУ ВО «ГУМРФ имени адмирала
С. О. Макарова»
198035, Российская Федерация,
Санкт-Петербург, ул. Двинская, 5/7
e-mail: *BarychnikovSO@gumrf.ru*
**Сахаров Владимир Васильевич** —
доктор технических наук, профессор
ФГБОУ ВО «ГУМРФ
имени адмирала С. О. Макарова»
198035, Российская Федерация,
Санкт-Петербург, ул. Двинская, 5/7
e-mail: *saharov_@rambler.ru*
**Чертков Александр Александрович** —
доктор технических наук, доцент
ФГБОУ ВО «ГУМРФ имени адмирала
С. О. Макарова»
198035, Российская Федерация, Санкт-Петербург,
ул. Двинская 5/7
e-mail: *chertkov51@mail.ru*,
*kaf _electroprivod@gumrf.ru.ru*



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